

The Menger and projective Menger properties of function spaces with the set-open topology

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Abstract

For a Tychonoff space X and a family λ of subsets of X , we denote by $C_\lambda(X)$ the space of all real-valued continuous functions on X with the set-open topology. In this paper, we study the Menger and projective Menger properties of a Hausdorff space $C_\lambda(X)$. Our main results state that

if λ is a π -network of X , then

(1) $C_\lambda(X)$ is Menger space, if and only, if $C_\lambda(X)$ is σ -compact,

and, if Y is a dense subset of X , then

(2) $C_p(Y|X)$ is projective Menger space, if and only, if $C_p(Y|X)$ is σ -pseudocompact.

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1. Introduction

Throughout this paper X will be a Tychonoff space. Let λ be a family non-empty subsets of X , $C(X)$ the set of all continuous real-valued function on X . Denote by $C_\lambda(X)$ the set $C(X)$ is endowed with the λ -open topology. The elements of the standard subbases of the λ -open topology will be denoted as follows: $[F, U] = \{f \in C(X) : f(F) \subseteq U\}$, where $F \in \lambda$, U is an open subset of \mathbb{R} . Note that if λ consists of all finite subsets of X then the λ -open topology is equal to the topology of pointwise convergence, that

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is $C_\lambda(X) = C_p(X)$. Denote by $C_p(Y|X) = \{h \in C_p(Y) : h = f|_Y \text{ for } f \in C(X)\}$ for $Y \subset X$.

Recall that, if X is a space and \mathcal{P} a topological property, we say that X is σ - \mathcal{P} if X is the countable union of subspaces with the property \mathcal{P} .

So a space X is called σ -compact (σ -pseudocompact, σ -bounded), if $X = \bigcup_{i=1}^{\infty} X_i$, where X_i is a compact (pseudocompact, bounded) for every $i \in \mathbb{N}$.

N.V. Velichko proved that $C_p(X)$ is σ -compact, if and only, if X is finite. In [20], V.V. Tkachuk clarified when $C_p(X)$ is σ -pseudocompact and when $C_p(X)$ is σ -bounded, and considered similar questions for the space $C_p^*(X)$ of bounded continuous functions on X .

A space X is said to be Menger [9] (or, [17]) if for every sequence $\{\mathcal{U}_n : n \in \omega\}$ of open covers of X , there are finite subfamilies $\mathcal{V}_n \subset \mathcal{U}_n$ such that $\bigcup \{\mathcal{V}_n : n \in \omega\}$ is a cover of X .

Every σ -compact space is Menger, and a Menger space is Lindelöf. The Menger property is closed hereditary, and it is preserved by continuous maps. It is well known that the Baire space $\mathbb{N}^{\mathbb{N}}$ (hence, \mathbb{R}^{ω}) is not Menger.

In [2], A.V. Arhangel'skii proved that $C_p(X)$ is Menger, if and only, if X is finite.

Let \mathcal{P} be a topological property. A.V. Arhangel'skii calls X *projectively* \mathcal{P} if every second countable image of X is \mathcal{P} . Arhangel'skii consider projective \mathcal{P} for $\mathcal{P} = \sigma$ -compact, analytic [3], and other properties.

Lj.D.R. Kočinac characterized the classical covering properties of Menger, Rothberger, Hurewicz and Gerlits-Nagy in term of continuous images in \mathbb{R}^{ω} . The projective selection principles were introduced and first time considered in [11].

Every Menger space is projectively Menger. It is known (Theorem 2.2 in [11]) that a space is Menger, if and only, if it is Lindelöf and projectively Menger.

Characterizations of projectively Menger spaces X in terms a selection principle restricted to countable covers by cozero sets are given in [5].

In [16], M. Sakai proved that $C_p(X)$ is projectively Menger, if and only, if X is pseudocompact and b -discrete.

In this paper we study the Menger property of Hausdorff space $C_\lambda(X)$, and the projective Menger property of $C_p(Y|X)$ where Y is dense subset of X .

2. Main definitions and notation

Recall that a family λ of non-empty subsets of a topological space (X, τ) is called a π -network for X if for any nonempty open set $U \in \tau$ there exists $A \in \lambda$ such that $A \subset U$.

Throughout this paper, a family λ of nonempty subsets of the set X is a π -network. This condition is equivalent to the space $C_\lambda(X)$ being a Hausdorff space [12].

We will also need the following assertion [1], [4].

Proposition 2.1. *If $\mathbb{I}_\alpha = \mathbb{I} = [0, 1]$ for $\alpha \in A$ and Y is a subspace of the Tychonoff cube $\mathbb{I}^A = \prod\{\mathbb{I}_\alpha : \alpha \in A\}$ which, whatever the countable set $B \subset A$, projects under the canonical projection $\pi_B : \mathbb{I}^A \mapsto \mathbb{I}^B$ onto the whole cube $\mathbb{I}^B = \prod\{\mathbb{I}_\alpha : \alpha \in B\}$ of \mathbb{I}^A , then Y is pseudocompact.*

Theorem 2.2. (Nokhrin [12]) *For a Tychonoff space X the following statements are equivalent:*

1. $C_\lambda(X)$ is a σ -compact;
2. X is a pseudocompact, $D(X)$ is a dense C^* -embedded set in X and family λ consists of all finite subsets of $D(X)$, where $D(X)$ is an isolated points of X .

The closure of a set A will be denoted by \overline{A} (or $cl(A)$); the symbol \emptyset stands for the empty set. As usual, $f(A)$ and $f^{-1}(A)$ are the image and the complete preimage of the set A under the mapping f , respectively.

A subset A of a space X is said to be *bounded* in X if for every continuous function $f : X \mapsto \mathbb{R}$, $f|_A : A \mapsto \mathbb{R}$ is a bounded function. Every σ -bounded space is projectively Menger (Proposition 1.1 in [3]).

3. Main results

In order to prove the main theorem we need to prove some statements that we call Lemmas, but note their self-importance.

Recall that a space X is called basically disconnected ([8]), if every cozero-set has an open closure. Clearly, every basically disconnected (Tychonoff) space is zero-dimensional space.

Lemma 3.1. *If $C_\lambda(X)$ is Menger, then X is a basically disconnected space.*

Proof. Let $U \subseteq X$ be a cozero set in X . Claim that $\overline{U} = \text{Int}\overline{U}$. Suppose that $\overline{U} \setminus \text{Int}\overline{U} \neq \emptyset$. Since U is a cozero set, there are open sets U_n of X such that for each $n \in \mathbb{N}$, $\overline{U_n} \subseteq U_{n+1}$ and $\bigcup_{n=1}^{\infty} U_n = U$. For each $n, m \in \mathbb{N}$, we put $Z_{n,m} = \{f \in C_\lambda(X, [0, 1]) : f|(X \setminus \text{Int}\overline{U}) \equiv 0 \text{ and } f(U_n) \subset [\frac{1}{2^m}, 1]\}$.

Note that $Z_{n,m}$ is closed subset of $C_\lambda(X)$ for each $n, m \in \mathbb{N}$. Let $h \notin Z_{n,m}$.

If $x \in X \setminus \text{Int}\overline{U}$ such that $h(x) \neq 0$. Since λ is π -network of X , there is $A \in \lambda$ such that $A \subset h^{-1}(h(x) - \frac{|h(x)|}{2}, h(x) + \frac{|h(x)|}{2}) \cap \text{Int}(X \setminus \text{Int}\overline{U})$. Then $h \in [A, (h(x) - \frac{|h(x)|}{2}, h(x) + \frac{|h(x)|}{2})]$ and $[A, (h(x) - \frac{|h(x)|}{2}, h(x) + \frac{|h(x)|}{2})] \cap Z_{n,m} = \emptyset$.

If $x \in U_n$ and $h(x) \notin [\frac{1}{2^m}, 1]$. Let $d = \frac{\text{diam}(h(x), [\frac{1}{2^m}, 1])}{2}$. Since λ is a π -network of X , there is $A \in \lambda$ such that $A \subset h^{-1}((h(x) - d, h(x) + d) \cap U_n$. Then $h \in [A, (h(x) - d, h(x) + d)]$ and $[A, (h(x) - d, h(x) + d)] \cap Z_{n,m} = \emptyset$.

Assume that $\bigcap \{Z_{n,m} : n \in \mathbb{N}\} = \emptyset$ for all $m \in \mathbb{N}$. Using the Menger property of $C_\lambda(X)$, we can take some $\varphi \in \mathbb{N}^\mathbb{N}$ such that $\bigcap \{Z_{\varphi(m),m} : m \in \mathbb{N}\} = \emptyset$. For each $m \in \mathbb{N}$, take any $g_m \in C_\lambda(X)$ satisfying $g_m(X \setminus \text{Int}(\overline{U})) \equiv 0$ and $g_m(U_{\varphi(m)}) = \{1\}$. Let $g = \sum_{j=1}^{\infty} 2^{-j} g_j$. Then, $g \in C_\lambda(X)$ and $g(X \setminus \text{Int}(\overline{U})) \equiv 0$. Fix any $m \in \mathbb{N}$, $1 \leq k \leq \varphi(m)$ and $x \in U_k$. Then we have

$$g(x) = \sum_{j=1}^{\infty} 2^{-j} g_j(x) \geq 2^{-m} g_m(x) = 2^{-m}.$$

Hence, $g \in \bigcap \{Z_{\varphi(m),m} : m \in \mathbb{N}\}$. This is a contradiction. Thus, there is some $m \in \mathbb{N}$ such that $\bigcap \{Z_{n,m} : n \in \mathbb{N}\} \neq \emptyset$. Let $p \in \bigcap \{Z_{n,m} : n \in \mathbb{N}\}$. Then $p(U) \subset [\frac{1}{2^m}, 1]$ and $p|(X \setminus \text{Int}\overline{U}) \equiv 0$. It follows that $\overline{U} \setminus \text{Int}\overline{U} = \emptyset$. \square

A subset $G \subset \omega^\omega$ is *dominating* if for every $f \in \omega^\omega$ there is a $g \in G$ such that $f(n) \leq g(n)$ for all but finitely many n .

Theorem 3.2. (Hurewicz [10]) *A second countable space X is Menger iff for every continuous mapping $f : X \mapsto \mathbb{R}^\omega$, $f(X)$ is not dominating.*

"Second countable" can be extended to "Lindelöf":

Theorem 3.3. (Kočinac [11], Theorem 2.2) *A Lindelöf space X is Menger iff for every continuous mapping $f : X \mapsto \mathbb{R}^\omega$, $f(X)$ is not dominating.*

Lemma 3.4. *If $C_\lambda(X)$ is Menger. Then X is pseudocompact.*

Proof. Assume that X is not pseudocompact and $f \in C(X)$ is not bounded function. Without loss of generality we can assume that $\mathbb{N} \subset f(X)$. For each $n \in \mathbb{N}$ we choose $A_n \in \lambda$ such that $A_n \subset f^{-1}((n - \frac{1}{3}, n + \frac{1}{3}))$. By

Lemma 3.1, $F_n = \overline{f^{-1}((n - \frac{1}{3}, n + \frac{1}{3}))}$ is clopen set for each $n \in \mathbb{N}$. Let $K = \{f \in C(X) : f|_{F_n} \equiv s_{f,n} \text{ for each } n \in \mathbb{N} \text{ and } s_{f,n} \in \mathbb{R}\}$. Then K is closed subset of $C_\lambda(X)$ and, hence, it is Menger. Fix $a_n \in A_n$ for every $n \in \mathbb{N}$. Note that $D = \{a_n : n \in \mathbb{N}\}$ is a C -embedded copy of \mathbb{N} (3L (1) in [8]). So we have a continuous mapping $F : K \mapsto \mathbb{R}^D$ the space K onto \mathbb{R}^D . But $F(K) = \mathbb{R}^D = \mathbb{R}^\omega$ is dominating, contrary to the Theorem 3.3. \square

Lemma 3.5. *If $C_\lambda(X)$ is Menger, then $\mu = \{A \in \lambda : A \text{ is finite subset of } X\}$ is a π -network of X .*

Proof. Assume that there exist an open set U of X such that $B \not\subset U$ for every $B \in \mu$. Fix a family $\{V_n : n \in \mathbb{N}\}$ of open subsets of X such that $V_n \subset U$ for every $n \in \mathbb{N}$ and $V_{n'} \cap V_{n''} = \emptyset$ for $n' \neq n''$. Fix $x_n \in V_n$ and $\epsilon > 0$. For every $f \in C_\lambda(X)$ and $n \in \mathbb{N}$ consider $B_{f,n} \in \lambda$ such that $B_{f,n} \subset f^{-1}((f(x_n) - \epsilon, f(x_n) + \epsilon)) \cap V_n$. Then $\mathcal{U}_n = \{[B_{f,n}, (f(x_n) - \epsilon, f(x_n) + \epsilon)] : f \in C_\lambda(X)\}$ is an open cover of $C_\lambda(X)$ for every $n \in \mathbb{N}$. Using the Menger property of $C_\lambda(X)$, for sequence $\{\mathcal{U}_n : n \in \omega\}$ of open covers of $C_\lambda(X)$, there are finite subfamilies $\mathcal{S}_n \subset \mathcal{U}_n$ such that $\bigcup \{\mathcal{S}_n : n \in \omega\}$ is a cover of $C_\lambda(X)$. Let $\mathcal{S}_n = \{[B_{f_{1,n}}, (f_{1,n}(x_n) - \epsilon, f_{1,n}(x_n) + \epsilon)], \dots, [B_{f_{k(n),n}}, (f_{k(n),n}(x_n) - \epsilon, f_{k(n),n}(x_n) + \epsilon)]\}$ for every $n \in \mathbb{N}$. Since $B_{f_s,n}$ is an infinite subset of X , we fix $z_{s,n} \in B_{f_s,n}$ for every $s \in \overline{1, k(n)}$ and $n \in \mathbb{N}$ such that $z_{s',n} \neq z_{s'',n}$ for $s' \neq s''$. Let $Z = \{z_{s,n} : s \in \overline{1, k(n)} \text{ and } n \in \mathbb{N}\}$.

Define the function $q : Z \mapsto \mathbb{R}$ such that $q(z_{s,n}) = 0$ if $0 \notin (f_{s,n}(x_n) - \epsilon, f_{s,n}(x_n) + \epsilon)$, else $q(z_{s,n}) = 2\epsilon$ for $s \in \overline{1, k(n)}$ and $n \in \mathbb{N}$. By Lemma 3.1, X is a basically disconnected space.

Recall that (14N p.215 in [8]) every countable set in a basically disconnected space is C^* -embedded.

Hence, there is $t \in C_\lambda(X)$ such that $t|_Z = q$. But $t \notin \bigcup \{\mathcal{S}_n : n \in \omega\}$. This is a contradiction. \square

Denote $D(X)$ a set of isolated points of X .

Lemma 3.6. *If $C_\lambda(X)$ is Menger, then $D(X)$ is dense set in X .*

Proof. Assume that there exist an open set $W \neq \emptyset$ such that $W \cap D(X) = \emptyset$. By Lemma 3.5, μ is π -network of X , hence, there is $A \in \mu$ such that $A \subset W$. Note that $X \setminus A$ is dense set in X . The constant zero function defined on

X is denoted by f_0 . For every $f \in C(X) \setminus \{f_0\}$ there is $x_f \in X \setminus A$ such that $f(x_f) \neq 0$. For every $f \in C(X) \setminus \{f_0\}$, consider $B_f \in \mu$ such that $B_f \subset f^{-1}((f(x_f) - \frac{|f(x_f)|}{2}, f(x_f) + \frac{|f(x_f)|}{2})) \cap (X \setminus A)$. Let $\epsilon > 0$. Then $\mathcal{V} = \{[B_f, (f(x_f) - \frac{|f(x_f)|}{2}, f(x_f) + \frac{|f(x_f)|}{2})] : f \in C(X) \setminus \{f_0\}\} \cup [A, (-\epsilon, \epsilon)]$ is an open cover of $C_\lambda(X)$. Since $C_\lambda(X)$ is Menger and, hence, $C_\lambda(X)$ is Lindelöf, there is a countable subcover $\mathcal{V}' = \{[B_{f_n}, (f_n(x_f) - \frac{|f_n(x_f)|}{2}, f_n(x_f) + \frac{|f_n(x_f)|}{2})] : \text{for } n \in \mathbb{N}\} \cup [A, (-\epsilon, \epsilon)] \subset \mathcal{V}$ of $C_\lambda(X)$. Since X is a basically disconnected space and every countable set in a basically disconnected space is C^* -embedded, there is $h \in C(X)$ such that $h|_{\bigcup_{n \in \mathbb{N}} B_{f_n}} \equiv 0$ and $h(a) = \epsilon$ for some $a \in A$. Note that $h \notin \bigcup \mathcal{V}'$, to contradiction. \square

Lemma 3.7. *If $C_\lambda(X)$ is Menger, then $D(X)$ is C^* -embedded.*

Proof. Let f be a bounded continuous function from $D(X)$ into \mathbb{R} , and $F_A = \{g \in C(X) : g|_A = f|_A\}$ for $A \in D(X)^\omega$. Note that F_A is closed subset of $C_\lambda(X)$ and, by Lemma 3.1, $F_A \neq \emptyset$. So $\xi = \{F_A : A \in D(X)^\omega\}$ is family of closed subspaces with the countable intersection property. Since $C_\lambda(X)$ is Menger, hence, it is Lindelöf, and every family of closed subspaces of with the countable intersection property has non-empty intersection. It follows that $\bigcap \xi \neq \emptyset$. We thus get that $\tilde{f} \in \bigcap \xi$ such that $\tilde{f} \in C(X)$ and $\tilde{f}|_{D(X)} = f$. \square

Proposition 3.8. *Let $X = \mathbb{N}$ and let $\lambda = \{X\} \cup \{\{x\} : x \in X\}$. Then $C_\lambda^*(X)$ is not Menger.*

Proof. Assume that $C_\lambda^*(X)$ is Menger. For every $i \in \mathbb{N}$ consider an open cover $\mathcal{V}_i = \{[\mathbb{N}, (-2 + \frac{1}{i+1}, 2 - \frac{1}{i+1})]\} \cup \{[x, (-\infty, -2 + \frac{2i+1}{2i(i+1)}) \cup (2 - \frac{2i+1}{2i(i+1)}, +\infty)] : x \in X\}$ of $C_\lambda^*(X)$. Using the Menger property of $C_\lambda^*(X)$, for sequence $\{\mathcal{V}_i : i \in \mathbb{N}\}$ of open covers of $C_\lambda^*(X)$, there are finite subfamilies $\mathcal{S}_i \subset \mathcal{V}_i$ such that $\bigcup \{\mathcal{S}_i : i \in \mathbb{N}\}$ is a cover of $C_\lambda^*(X)$.

Without loss of generality we can assume that $[\mathbb{N}, (-2 + \frac{1}{i+1}, 2 - \frac{1}{i+1})] \in \mathcal{S}_i$ for each $i \in \mathbb{N}$.

By using induction, for each $i \in \mathbb{N}$, determine the values of the function f at some points, depending on the \mathcal{S}_i , as follows:

for $i = 1$ and

$\mathcal{S}_1 = \{[\mathbb{N}, (-2 + \frac{1}{2}, 2 - \frac{1}{2})], [x_1^1, (-\infty, -2 + \frac{3}{4}) \cup (2 - \frac{3}{4}, +\infty)], \dots, [x_k^1, (-\infty, -2 + \frac{3}{4}) \cup (2 - \frac{3}{4}, +\infty)]\}$, define

$f(x_n^1) = 0$ for $n \in \overline{1, k}$ and
 $f(s_1) = p_1$ where $p_1 \in [-2 + \frac{5}{12}, 2 - \frac{5}{12}] \setminus (-2 + \frac{1}{2}, 2 - \frac{1}{2})$ for some $s_1 \in X \setminus \{x_n^1 : n \in \overline{1, k}\}$. Denote $P_1 = \bigcup_{n \in \overline{1, k}} x_n^1 \cup s_1$.

for $i = m$
 $\mathcal{S}_m = \{[\mathbb{N}, (-2 + \frac{1}{m+1}, 2 - \frac{1}{m+1})], [x_1^m, (-\infty, -2 + \frac{2m+1}{2m(m+1)}) \cup (2 - \frac{2m+1}{2m(m+1)}, +\infty)], \dots, [x_{k(m)}^m, (-\infty, -2 + \frac{2m+1}{2m(m+1)}) \cup (2 - \frac{2m+1}{2m(m+1)}, +\infty)]\}$, define
 $f(x_n^m) = 0$ where $x_n^m \notin P_{m-1}$ for $n \in \overline{1, k(m)}$ and
 $f(s_m) = p_m$ where $p_m \in [-2 + \frac{2(m+1)+1}{2(m+1)(m+2)}, 2 - \frac{2(m+1)+1}{2(m+1)(m+2)}] \setminus (-2 + \frac{1}{m+1}, 2 - \frac{1}{m+1})$ for some $s_m \in X \setminus P_{m-1}$. Denote $P_m = \bigcup_{n \in \overline{1, k(m)}} x_n^m \cup s_m \cup P_{m-1}$ and

$$P = \bigcup_{m \in \mathbb{N}} P_m.$$

If $X \setminus P \neq \emptyset$, then let $f(x) = 1$ for $x \in X \setminus P$.

By construction of f , $f \notin \mathcal{S}_i$ for every $i \in \mathbb{N}$, to contradiction. \square

Lemma 3.9. *If $C_\lambda(X)$ is Menger, then each $A \in \lambda$ is finite subset of $D(X)$.*

Proof. Suppose that $C_\lambda(X)$ is Menger, $\tilde{\lambda} = \{A\} \cup \{\{x\}, x \in D(X)\}$ and $A \in \lambda$ is an infinite subset of X . Then $C_{\tilde{\lambda}}(X)$ is Menger, too. Note that if A is countable and $A \subset D(X)$, then we have a continuous mapping $g : C_{\tilde{\lambda}}(X) \mapsto C_{p \cup \{\mathbb{N}\}}^*(\mathbb{N})$. Hence, $C_{p \cup \{\mathbb{N}\}}^*(\mathbb{N})$ is Menger, contrary to Proposition 3.8.

Let $V = (-1, 1) \cup (\mathbb{R} \setminus [-4, 4])$. Consider $\mathcal{U} = \{[A, V]\} \cup \{[x, \mathbb{R} \setminus [-\frac{2}{3}, \frac{2}{3}]] : x \in D(X)\}$. Since $D(X)$ is dense subset of $C_{\tilde{\lambda}}(X)$ (Lemma 3.6), \mathcal{U} is an open cover of $C_{\tilde{\lambda}}(X)$ and, hence, there is a countable subcover $\mathcal{U}' \subset \mathcal{U}$ of $C_{\tilde{\lambda}}(X)$. Let $\mathcal{U}' = \{[A, V], [x_1, \mathbb{R} \setminus [-\frac{2}{3}, \frac{2}{3}]], \dots, [x_n, \mathbb{R} \setminus [-\frac{2}{3}, \frac{2}{3}]], \dots\}$. Let $z \in A \setminus \bigcup_{n \in \mathbb{N}} \{x_n\}$ (note that either $z \in A \setminus D(X)$ or $A \subset D(X)$ and $|A| > \aleph_0$). Since every countable set in a basically disconnected space is C^* -embedded, there is $h \in C_{\tilde{\lambda}}(X)$ such that $h|_{\bigcup_{n \in \mathbb{N}} \{x_n\}} = 0$ and $h(z) = 2$. It follows that $h \notin \bigcup \mathcal{U}'$, to contradiction. It follows that A is finite subset of $D(X)$. \square

Theorem 3.10. *Let X be a Tychonoff space and let λ be a π -network of X . Then a space $C_\lambda(X)$ is Menger, if and only if, $C_\lambda(X)$ is σ -compact.*

Proof. By Lemma 3.4, X is pseudocompact. By Lemmas 3.5 and 3.9, the family λ consists of all finite subsets of $D(X)$, where $D(X)$ is an isolated

points of X . By Lemma 3.7, $D(X)$ is a dense C^* -embedded set in X . It follows that $C_\lambda(X)$ is σ -compact (Theorem 2.2). \square

Various properties between σ -compactness and Menger are investigated in the papers [19, 6]. We can summarize the relationships between considered notions in ([19], see Figure 1), Theorems 3.10 and 2.2. Then we have the next

Theorem 3.11. *For a Tychonoff space X and a π -network λ of X , the following statements are equivalent:*

1. $C_\lambda(X)$ is σ -compact;
2. $C_\lambda(X)$ is Alster;
3. (CH) $C_\lambda(X)$ is productively Lindelöf;
4. "TWO wins M -game" for $C_\lambda(X)$;
5. $C_\lambda(X)$ is projectively σ -compact and Lindelöf;
6. $C_\lambda(X)$ is Hurewicz;
7. $C_\lambda(X)$ is Menger;
8. X is a pseudocompact, $D(X)$ is a dense C^* -embedded set in X and family λ consists of all finite subsets of $D(X)$, where $D(X)$ is an isolated points of X .

4. Projectively Menger space

According to Tkačuk [20], a space X said to be *b-discrete* if every countable subset of X is closed (equivalently, closed and discrete) and C^* -embedded in X .

Lemma 4.1. *(Lemma 2.1 in [16]) The following are equivalent for a space X :*

1. X is *b-discrete*;
2. For any disjoint countable subsets A and B in X , there are disjoint zero-sets Z_A and Z_B in X such that $A \subset Z_A$ and $B \subset Z_B$;
3. For any disjoint countably subsets A and B in X such that A is closed in X , there are disjoint zero-sets Z_A and Z_B in X such that $A \subset Z_A$ and $B \subset Z_B$.

Definition 4.2. For $A \subset X$, a space X will be called b_A -discrete if every countable subset of A is closed in A and C^* -embedded in X .

Lemma 4.3. *The following are equivalent for a space X and $A \subset X$:*

1. X is b_A -discrete;
2. For any disjoint countable subsets D and B in A , there are disjoint zero-sets Z_D and Z_B in X such that $D \subset Z_D$ and $B \subset Z_B$;
3. For any disjoint countably subsets D and B in A such that D is closed in A , there are disjoint zero-sets Z_A and Z_B in X such that $D \subset Z_D$ and $B \subset Z_B$.

Similarly to the proof of implication $(C_p(X, \mathbb{I}) \text{ is projectively Menger} \Rightarrow X \text{ is } b\text{-discrete})$ of Theorem 2.4 in [16], we claim the next

Lemma 4.4. *Let $C_\lambda(X)$ be a projectively Menger space, then X is b_A -discrete where $A = \bigcup \lambda$.*

Proof. Let $C_\lambda(X)$ be a projectively Menger. We show the statement (3) in Lemma 4.3. Let D and B be a disjoint countable subsets in A such that D is closed in A . Let $B = \{b_n : n \in \mathbb{N}\}$, and let $B_n = \{b_1, \dots, b_n\}$.

For each $n, m \in \mathbb{N}$, we put $Z_{n,m} = \{f \in C_\lambda(X) : f(D) = \{0\} \text{ and } f(B_m) \subset [\frac{1}{2^n}, 1]\}$. Since D and B_m are countable and λ is a π -network of X , each $Z_{n,m}$ is a zero-set in $C_\lambda(X)$. Assume that $\bigcap \{Z_{n,m} : m \in \mathbb{N}\} = \emptyset$ for all $n \in \mathbb{N}$. Using the projective Menger property of $C_\lambda(X)$, Theorem 6 in [5], we can take some $\varphi \in \mathbb{N}^{\mathbb{N}}$ such that $\bigcap \{Z_{n,\varphi(n)} : n \in \mathbb{N}\} = \emptyset$. For each $n \in \mathbb{N}$, take any $g_n \in C_\lambda(X)$ satisfying $g_n(D) = \{0\}$ and $g_n(B_{\varphi(n)}) = \{1\}$. Let $g = \sum_{j=1}^{\infty} 2^{-j} g_j$. Then, $g \in C_\lambda(X)$ and $g(D) \equiv 0$. Fix any $n \in \mathbb{N}$,

$1 \leq k \leq \varphi(m)$. Then we have

$$g(b_k) = \sum_{j=1}^{\infty} 2^{-j} g_j(b_k) \geq 2^{-n} g_n(b_k) = 2^{-n}.$$

Hence, $g \in \bigcap \{Z_{n,\varphi(n)} : n \in \mathbb{N}\}$. This is a contradiction. Thus, there is some $n \in \mathbb{N}$ such that $\bigcap \{Z_{n,m} : m \in \mathbb{N}\} \neq \emptyset$. Let $h \in \bigcap \{Z_{n,m} : m \in \mathbb{N}\}$. Then $D \subset Z_A = h^{-1}(0)$ and $B \subset Z_B = h^{-1}([\frac{1}{2^n}, 1])$. □

Theorem 4.5. *Let X be a Tychonoff space and let Y be a dense subset of X . Then the following statements are equivalent:*

1. $C_p(Y|X)$ is projectively Menger;

2. $C_p(Y|X)$ is σ -bounded;
3. $C_p(Y|X)$ is σ -pseudocompact;
4. X is pseudocompact and b_Y -discrete.

Proof. Note that $C_p(Y|X)$ is homeomorphic to $C_\lambda(X)$ for $\lambda = [Y]^{<\omega}$.

(1) \Rightarrow (4). By Lemma 4.4, X is b_Y -discrete. Assume that X is not pseudocompact and $f \in C(X)$ is not bounded function. Without loss of generality we can assume that $\mathbb{N} \subset f(X)$. For each $n \in \mathbb{N}$ we choose $a_n \in Y$ such that $a_n \in f^{-1}((n - \frac{1}{3}, n + \frac{1}{3}))$. Note that $D = \{a_n : n \in \mathbb{N}\}$ is a C -embedded copy of \mathbb{N} (3L (1) in [8]). So we have a continuous mapping $F : C_p(Y|X) \mapsto \mathbb{R}^D$ the Menger space $C_p(Y|X)$ onto \mathbb{R}^D . But $F(C_p(Y|X)) = \mathbb{R}^D = \mathbb{R}^\omega$ is dominating, contrary to the Theorem 3.3.

(4) \Rightarrow (3). Since $C_p(Y|X, \mathbb{I})$ is a dense subset of \mathbb{I}^Y and X is b_Y -discrete, by Proposition 2.1, $C_p(Y|X, \mathbb{I})$ is pseudocompact. Hence, $C_p(Y|X)$ is σ -pseudocompact.

Note that every σ -pseudocompact space is σ -bounded, and every σ -bounded space is projectively Menger (Proposition 1.1 in [3]). \square

5. Examples

Using Theorem 3.10 and Theorem 4.5, we can construct example of projective Menger topological group $C_\lambda(X)$ such that it is not Menger.

Note that if $\lambda = [\bigcup \lambda]^{<\omega}$, then $C_\lambda(X)$ is a topological group (locally convex topological vector space, topological algebra) ([14], [15]).

Example 5.1. (*Example 1 in [13]*) Let T be a P -space without isolated points, $X = \beta(T)$ and let λ be a family of all finite subsets of T . Then $C_\lambda(X)$ is σ -countably compact (Theorem 1.2 in [13]), hence, the topological group $C_\lambda(X)$ is projective Menger. But the space X does not contain isolated points, hence, $C_\lambda(X)$ is not Menger.

Example 5.2. (*Example 2 in [13]*) Let D be an uncountable discrete space and $\lambda = D^{<\omega}$. Consider $F = \beta(D) \setminus \bigcup \{\overline{S} : S \subset D, \text{ and } S \text{ countable}\}$. Denote by $b(D)$ a quotient space obtained from $\beta(D)$ by identifying the set F with the point $\{F\}$. Then the topological group $C_\lambda(b(D))$ is projective Menger (σ -countably compact), but is not Menger.

Example 5.3. ([18]) D.B.Shahmatov has constructed for an arbitrary cardinal $\tau \geq 2^{\aleph_0}$ an everywhere dense pseudocompact space X_τ in \mathbb{I}^τ such that

X_τ is a b -discrete. Hence, the topological group $C_p(X_\tau)$ is projective Menger (σ -pseudocompact and is not σ -countably compact), but is not Menger for an arbitrary cardinal $\tau \geq 2^{\aleph_0}$.

Remark 5.4. By Theorems 2.2 and 3.10, if X is compact, λ is a π -network of X and $C_\lambda(X)$ is Menger, then X is homeomorphic to $\beta(D)$, where $\beta(D)$ is Stone-Čech compactification of a discrete space D , and $\lambda = [D]^{<\omega}$.

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